Two phase free boundary problems

Domenico Angelo La Manna
(in collaboration with S. Guarino Lo Bianco and B. Velichkov)

domenicoangelo.lamanna@unina.it

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Heuristic

Assume we are given ice and put in a container, say D and we put that into water. We know that part of the container will stay outside the water and part of it will stay inside. There will be exchange of energy between the environment and the container and in fact some heat will be transferred to the ice which starts to melt. After some time, inside the container there will be two phases, a liquid one and a solid one. We want to understand the heat distribution and some properties of the free surface, i.e. the contact surface between the liquid phase and the solid phase.

Mathematical setting

Let $A \subset \mathbb{R}^d$ be a an open set in $\mathbb{R}^d$. We recall that the set $E \subset \mathbb{R}^d$ is said to have a *finite perimeter in A* if

$$\mathrm{Per}(E, A) = \sup \left\{ \int_A \operatorname{div} \chi(x)\, dx \ : \ \chi \in C^1_c(A; \mathbb{R}^d), \ \sup_{x \in \mathbb{R}^d} |\chi(x)| \leq 1 \right\},$$

is finite. We define the capacity (or the 2-capacity) of a set $E \subset \mathbb{R}^d$ as

$$\mathrm{cap}(E) = \inf \left\{ \|u\|^2_{H^1(\mathbb{R}^d)} \ : \ u \in H^1(\mathbb{R}^d), \ u \geq 1 \text{ in a neighborhood of } E \right\}.$$

Suppose now that $d \geq 3$. It is well-known that the sets of zero capacity have zero $d-1$ dimensional Hausdorff measure

$$\text{If} \quad \mathrm{cap}(E) = 0\ , \quad \text{then} \quad \mathcal{H}^{d-1}(E) = 0.$$

Sobolev functions

The Sobolev functions are defined up to a set of zero capacity (i.e. *quasi-everywhere*), that is, if $A \subset \mathbb{R}^d$ is an open set and $u \in H^1(A)$, then there is a set $\mathcal{N}_u \subset \mathbb{R}^d$ such that $\operatorname{cap}(\mathcal{N}_u) = 0$ and

$$u(x_0) = \lim_{r \to 0} \frac{1}{|B_r|} \int_{B_r(x_0)} u(x)\, dx \qquad \text{for every} \qquad x_0 \in A \setminus \mathcal{N}_u.$$

Moreover, for every function $u \in H^1(A)$ there is a sequence $u_n \in C^\infty(A) \cap H^1(A)$ and a set $\mathcal{N} \subset A$ of zero capacity such that:

- u_n converges to u strongly in $H^1(A)$;
- $u(x) = \lim_{n\to\infty} u_n(x)$ for every $x \in A \setminus (\mathcal{N} \cup \mathcal{N}_u)$.

Sobolev

More over we have that there is a set $\mathcal{N}_u \subset D$ such that $\mathcal{H}^{d-1}(\mathcal{N}_u) = 0$ and

$$u(x_0) = \lim_{r \to 0} \frac{1}{|B_r|} \int_{B_r(x_0)} u(x)\, dx \qquad \text{for every} \qquad x_0 \in D \setminus \mathcal{N}_u.$$

In particular, if $E \subset \mathbb{R}^d$ is a set of locally finite perimeter in the open set $A \subset \mathbb{R}^d$ and if $u \in H^1(A)$, then the function u^2 is defined $\mathcal{H}^{d-1}$-almost everywhere on $\partial^* E$ and is $\mathcal{H}^{d-1}$ measurable on $\partial^* E$. Thus, the integral

$$\mathcal{I}(u, E) := \int_{A \cap \partial^* E} u^2 \, d\mathcal{H}^{d-1} \quad \text{is well-defined.}$$

An insulation problem

Let $E \subset \mathbb{R}^d$ and $v \in H^1(\mathbb{R}^d)$ with $v \geq m > 0$ and

$$\int_{\partial^* E} v^2 d\mathcal{H}^{d-1} < \infty.$$

Define the admissible sets

$$\mathcal{V} = \{u \in H^1_{loc}(\mathbb{R}^d) \ : \ u - v \in H^1_0(D)\},$$

$$\mathcal{E} = \{\Omega \subset \mathbb{R}^d \ : \ \mathrm{Per}(\Omega) < +\infty \ \text{ and } \ \Omega = E \ \text{ in } \ \mathbb{R}^d \setminus D\},$$

and we consider the variational minimization problem

$$\min\{J_\beta(u, \Omega) \ : \ u \in \mathcal{V}, \ \Omega \in \mathcal{E}\}. \tag{1.1}$$

where

$$J_\beta(u, \Omega) = \int_D |\nabla u|^2 \, dx + \int_{\partial^* \Omega} u^2 d\,\mathcal{H}^{n-1}$$

Main theorem

Theorem 1 (Existence and regularity of minimizers)

Let $\beta > 0$, $D \subset \mathbb{R}^d$, v, E, $\mathcal{V}$ and $\mathcal{E}$ be as above. Then the following holds.

1. *There exists a solution $(u, \Omega) \in \mathcal{V} \times \mathcal{E}$ to the variational problem* (1.1).
2. *For every solution (u, Ω) of* (1.1), *u is Hölder continuous and bounded from below by a strictly positive constant in D.*
3. *If (u, Ω) is a solution to* (1.1), *then the free boundary $\partial\Omega \cap D$ can be decomposed as the disjoint union of a regular part $\mathrm{Reg}(\partial\Omega)$ and a singular part $\mathrm{Sing}(\partial\Omega)$, where :*
   - $\mathrm{Reg}(\partial\Omega)$ *is a C^∞ hypersurface and a relatively open subset of $\partial\Omega$, and the function u is C^∞ smooth on $\mathrm{Reg}(\partial\Omega)$;*
   - $\mathrm{Sing}(\partial\Omega)$ *is a closed set, which is empty if $d \leq 7$, discrete if $d = 8$, and of Hausdorff dimension $d - 8$, if $d > 8$.*

Consequences

We notice that if (u, Ω) is a solution to (1.1), then u is harmonic in the interior of Ω and $D \setminus \Omega$. Thus, as a consequence of Theorem 2 (iii), in a neighborhood of a regular point $x_0 \in \mathrm{Reg}(\partial\Omega)$, the functions $u : \overline{\Omega} \to \mathbb{R}$ and $u : \overline{D \setminus \Omega} \to \mathbb{R}$ are C^∞ up to the free boundary $\partial\Omega$. To be more precise, the Euler-Lagrange equation is given by the system

$$\begin{cases} \Delta u_+ = 0 \quad \text{in} \quad \Omega, \\ \Delta u_- = 0 \quad \text{in} \quad D \setminus \overline{\Omega}, \\ u_+ = u_- = u \quad \text{on} \quad \partial\Omega, \\ \dfrac{\partial u_+}{\partial \nu_\Omega} - \dfrac{\partial u_-}{\partial \nu_\Omega} + 2\beta u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

Existence of a solution

The main difficulty in order to prove that theorem comes from proving the existence of a solution. In fact, for a minimizing sequence (u_n, Ω_n) we have

$$\int_D |\nabla u_n|^2 \, dx + \int_{\partial \Omega_n} u_n^2 \, d\mathcal{H}^{d-1} \leq C$$

which readly implies the existence of a function u_∞ such that $u_n \to u_\infty$ weak in H^1. We also know that $\int_D |\nabla u|^2 \, dx$ is semicontinuous with respect to the weak H^1 convergence so the first addend is fine. So, the next thing to prove is semicontinouity of the second addend of the functional.

Semicontinuoity

Lemma 1

Suppose that $A \subset \mathbb{R}^d$. Let $u_n \in H^1(A) \cap L^\infty(D)$ be a sequence of non negative functions and $\Omega_n \subset \mathbb{R}^d$ be a sequence of sets of locally finite perimeter in A such that:

1. *there is a function $u_\infty \in H^1(A)$ such that u_n converges to u_∞ weakly in $H^1(A)$ and pointwise almost-everywhere in A;*
2. *there is a set $\Omega_\infty \subset \mathbb{R}^d$ of locally finite finite perimeter in A such that the sequence of characteristic functions χ_{Ω_n} converges to χ_{Ω_∞} pointwise almost-everywhere in A.*

Then,

$$\int_{A\cap\partial^*\Omega_\infty} u_\infty^2 \, d\mathcal{H}^{d-1} \le \liminf_{n\to\infty} \int_{A\cap\partial^*\Omega_n} u_n^2 \, d\mathcal{H}^{d-1}. \tag{1.2}$$

Existence of a solution: main problem

The main problem now is that for a generic minimizing sequence we are not able to extract a subsequence such that $P(\Omega_n) \leq C_1$ for some constant C_1.
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How do we solve this problem?
The trick now is to select a better minimizing sequence via approximation of the problem. For $\varepsilon > 0$ we define the ε approximated problem

$$\inf \left\{ J_\beta(\Omega, u) \ : \ \Omega \in \mathcal{E}, \ u \in \mathcal{V}, \ u \geq \varepsilon \text{ in } \mathbb{R}^d \right\}. \tag{1.3}$$

Approximated problem: existence and regularity

Lemma 2

Let $m > 0$, $\beta > 0$ and $\varepsilon \in [0, m)$ be fixed. Let the function $u_\varepsilon \in H^1(D)$ and the set of finite perimeter Ω_ε be such that the couple $(u_\varepsilon, \Omega_\varepsilon)$ is a solution to the problem (1.3) with some $v \in H^1(D)$ and $E \subset \mathbb{R}^d$. Then, for every $\delta > 0$, there is a constant C depending on D, δ and v (but not on ε) such that

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C|x - y|^{\frac{1}{3}} \qquad \text{for every} \qquad x, y \in D_\delta.$$

In addiction, the sets Ω_ε are $C^{1,\alpha}$ up to an $n - 8$ dimensional set.

Construction of a solutiom: definition of u_0

Now, for any $\varepsilon \in (0,1)$, we consider the solution $(u_\varepsilon, \Omega_\varepsilon)$ of (1.3). As a consequence of this lemma we can find a sequence $\varepsilon_n \to 0$ and a function $u_0 \in H^1(D) \cap C^{0,\frac{1}{3}}(D)$ such that :

- u_{ε_n} converges to u_0 uniformly on every set D_δ, $\delta > 0$, where $D_\delta = \{x \in D : d(x, D^c) \geq \delta\}$
- u_{ε_n} converges to u_0 strongly in $L^2(D)$;
- u_{ε_n} converges to u_0 weakly in $H^1(D)$.

Our aim is to show that u_0 is a solution to (1.1).

Frame Title

Fix $t > 0$ and $\delta > 0$ and we notice that the perimeter of Ω_{ε_n} is bounded on the open set $\{u_0 > t\} \cap D_\delta$. Indeed, the uniform convergence of u_{ε_n} to u_0 implies that, for n large enough ($n \geq N_{t,\delta}$, for some fixed $N_{t,\delta} \in \mathbb{N}$),

$$u_{\varepsilon_n} \geq \frac{t}{2} \quad \text{on} \quad D_\delta \cap \{u_0 > t\}.$$

Thus, we have

$$J_\beta(v, E) \geq \beta \int_{D_\delta \cap \{u_0 > t\} \cap \partial^* \Omega_{\varepsilon_n}} u_{\varepsilon_n}^2 \, d\mathcal{H}^{d-1} \geq \frac{\beta t^2}{2} \operatorname{Per}\left(\Omega_{\varepsilon_n}; D_\delta \cap \{u_0 > t\}\right)$$

Hence this gives that for a.e $t > 0$ it holds

$$\operatorname{Per}\left(\Omega_{\varepsilon_n} \cap \{u_0 > t\} \cap D_\delta\right) \leq C_{t,\delta} \quad \text{for every} \quad n \geq N_{t,\delta},$$

for some constant $C_{t,\delta} > 0$.

Construction of Ω_0

Now, since all the sets $\Omega_{\varepsilon_n} \cap \{u_0 > t\} \cap D_\delta$ are contained in D and have uniformly bounded perimeter, we can find a set Ω_0 and a subsequence for which

$$\chi_{\Omega_{\varepsilon_n} \cap \{u_0 > t\} \cap D_\delta}(x) \to \chi_{\Omega_0 \cap \{u_0 > t\} \cap D_\delta}(x) \quad \text{for almost-every} \quad x \in D.$$

Thus, by a diagonal sequence argument, we can extract a subsequence of ε_n (still denoted by ε_n) and we can define the set $\Omega_0 \subset \mathbb{R}^d$ as the pointwise limit

$$\chi_{\Omega_0}(x) = \lim_{n \to \infty} \chi_{\Omega_{\varepsilon_n} \cap \{u_0 > 0\}}(x) \quad \text{for almost-every} \quad x \in \{u_0 > 0\},$$

and we notice that, by construction, $\Omega_0 \subset \{u_0 > 0\}$.

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and we notice that, by construction, $\Omega_0 \subset \{u_0 > 0\}$.
Notice that, we do not know a priori that Ω_0 has finite perimeter. We only know that

$$\mathrm{Per}\left(\Omega_0 \cap \{u_0 > t\} \cap D_\delta\right) < \infty \quad \forall \delta > 0 \quad \text{and almost-every} \quad t > 0$$

which means that $\Omega_0 \cap \{u_0 > t\}$ has locally finite perimeter in D

What is missing?

We have now constructed our canditate solution to the problem (u_0, Ω_0). To prove that this couple is a solution we need now to prove

- $J(u_0, \Omega_0) \leq J(v, \Omega)$ for all $(v, \Omega) \in \mathcal{A} \times \mathcal{V}$

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- $u_0 \in H^1(D)$ and $u_0 - v \in H^1_0(D)$
- Ω_0 has finite perimeter

An optimal condition

The next step to prove existence consists in the following

Lemma 3 (The optimality condition at the limit)

Let u_0 and Ω_0 be the constructed couple. Then, for almost-every $t > 0$, we have

$$\int_{\{u_0<t\}} |\nabla u_0|^2 \, dx \leq \beta \, t^2 \, \mathrm{Per}\big(\{u_0 < t\}\big). \tag{1.4}$$

To prove this lemma fix $t > 0$ such that the set $\{u_0 < t\}$ has finite perimeter. Then, for n large enough use the couple $(u_0 \vee t, \Omega_0 \cup \{u_0 < t\})$ to test the optimality of $(u_{\varepsilon_n}, \Omega_{\varepsilon_n})$. Notice that the set $\Omega_0 \cup \{u_0 < t\}$ has finite perimeter for a.e $t \in (0, m)$, thus this is an admissible competitor, and then send carefully $n \to \infty$ using the semicontinuoity lemma.

Non degeneracy

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Proposition 4 (Non-degeneracy)

Let $\beta > 0$, $m > 0$, D be a bounded open set of $\mathbb{R}^d$ and $u \in H^1(D)$ be a non-negative function in D such that $u \geq m$ on ∂D. Let $\Omega \subset D$ be a set of finite perimeter in D. Suppose that u and Ω satisfy the optimality condition

$$\int_{\Omega_t} |\nabla u|^2 \, dx \leq \beta \, t^2 \, \mathrm{Per}(\Omega_t) \qquad \text{where} \qquad \Omega_t = \{u \leq t\}, \quad (1.5)$$

for almost-every $t \in (0, m)$. Then, $|\Omega_t| = 0$ for some $t > 0$.

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for almost-every $t \in (0, m)$. Then, $|\Omega_t| = 0$ for some $t > 0$.

Proof of non-degeneracy

Step one.

By contradiction, suppose that

$$|\Omega_t| > 0 \quad \text{for every} \quad t > 0.$$

Let $t \in (0, m)$ be fixed. By the co-area formula, the Cauchy-Schwartz inequality and the optimality condition (1.5), we get

$$\int_{\Omega_t} |\nabla u| = \int_0^t \mathrm{Per}(\Omega_s)\, ds \le \left(\int_{\Omega_t} |\nabla u|^2 \right)^{\frac{1}{2}} |\Omega_t|^{\frac{1}{2}} \le t\beta^{\frac{1}{2}} \mathrm{Per}(\Omega_t)^{\frac{1}{2}} |\Omega_t|^{\frac{1}{2}}. \tag{1.6}$$

Set

$$f(t) := \int_0^t \mathrm{Per}(\Omega_s)\, ds = \int_{\Omega_t} |\nabla u|\, dx \ .$$

Note that $f(0) = 0$ since u is non negative

Proof of non-degeneracy

Step two.

By the isoperimetric inequality and the estimate (1.6), there is a dimensional constant C_d such that

$$\int_0^t \mathrm{Per}(\Omega_s)\, ds \leq t\beta^{\frac{1}{2}} C_d \,\mathrm{Per}(\Omega_t)^{\frac{2d-1}{2d-2}} .$$

Using the definition of f, we can re-write this inequality as

$$f(t)^{\frac{2d-2}{2d-1}} \leq t^{\frac{2d-2}{2d-1}} \left(\beta^{\frac{1}{2}} C_d\right)^{\frac{2d-2}{2d-1}} f'(t) .$$

After rearranging the terms and integrating from 0 to t, we obtain

$$f(t)^{\frac{1}{2d-1}} - f(0)^{\frac{1}{2d-1}} \geq \frac{t^{\frac{1}{2d-1}}}{\left(\beta^{\frac{1}{2}} C_d\right)^{\frac{2d-2}{2d-1}}} .$$

Thus we arrive at

$$f(t) \geq Ct.$$

Proof of non-degeneracy
Step three.

Let $\alpha \in (0,1)$ be fixed. Then, we have that

$$\int_0^t \mathrm{Per}(\Omega_s)^\alpha \, |\Omega_s|^{1-\alpha} ds \le \left(\int_0^t \mathrm{Per}(\Omega_s)\, ds \right)^\alpha \left(\int_0^t |\Omega_s|\, ds \right)^{1-\alpha}$$
$$\le \left(t\beta^{\frac{1}{2}} \mathrm{Per}(\Omega_t)^{\frac{1}{2}} |\Omega_t|^{\frac{1}{2}} \right)^\alpha \left(t|\Omega_t| \right)^{1-\alpha}$$
$$= t\beta^{\frac{\alpha}{2}} \mathrm{Per}(\Omega_t)^{\frac{\alpha}{2}} |\Omega_t|^{1-\frac{\alpha}{2}}.$$

Thus, we obtain that for fixed $T \in (0, m)$ and $C > 0$, the following implication holds :

$$\text{If} \quad C \le \mathrm{Per}(\Omega_t)^\alpha |\Omega_t|^{1-\alpha} \quad \text{for every} \quad t \in (0, T),$$
$$\text{then} \quad C \le \beta^{\frac{\alpha}{2}} \mathrm{Per}(\Omega_t)^{\frac{\alpha}{2}} |\Omega_t|^{1-\frac{\alpha}{2}} \quad \text{for every} \quad t \in (0, T). \tag{1.7}$$

Proof of non-degeneracy
Conclusion

Next we observe

$$\beta^{-d} C_d \leq |\Omega_t| \qquad \text{for every} \qquad t \in [0, m). \tag{1.8}$$

Indeed, using step two and arguing by induction we get for every $n \geq 1$ and every $t \in (0, m)$, we have the inequality

$$C \leq \beta^{1-\frac{1}{2^n}} \mathrm{Per}(\Omega_t)^{\frac{1}{2^n}} |\Omega_t|^{1-\frac{1}{2^n}}. \tag{1.9}$$

and then we can send $n \to \infty$. Thus if β is small enough (1.8) gives a contraddiction.

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and then we can send $n \to \infty$. Thus if β is small enough (1.8) gives a contraddiction.
To have the result for a general β we prove that for every $\varepsilon > 0$, there is T_ε such that for all $t \in (0, T_\varepsilon)$ we have

$$\int_{\Omega_t} |\nabla u| = \int_0^t \mathrm{Per}(\Omega_s)\, ds \leq t\varepsilon^{\frac{1}{2}} \mathrm{Per}(\Omega_t)^{\frac{1}{2}} |\Omega_t|^{\frac{1}{2}}. \tag{1.10}$$

Proof of the main Theorem

With the non-degeneracy lemma in our hands not it is easy to prove the existence of a minimizer. In fact, the non degeneracy lemma ensures us that $u_0 \geq \kappa > 0$ for some κ. The semicontinouity lemma (easy to use this time) gives that

$$J(u_0, \Omega_0) \leq J(v, \Omega)$$

for all $(v, \Omega) \in \mathcal{A} \times \mathcal{V}$. Since $u_0 \geq \kappa$ we now prove immediately that Ω_0 is a set of finite perimeter.

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for all $(v, \Omega) \in \mathcal{A} \times \mathcal{V}$. Since $u_0 \geq \kappa$ we now prove immediately that Ω_0 is a set of finite perimeter. To prove the higher regularity we prove that Ω_0 is a (Λ, r_0)-quasi minimizer of the perimeter and use the De Giorgi regularity theory to infer that Ω_0 is a $C^{1,\alpha}$ hypersurgace (up to a small set), then we use this information and the Euler equation togheter with a bootstrap argument to improve the regularity of Ω_0 (close to a regular point) and the regularity of u_0.

We study the problem of separation of two species (double phase) in case the interaction among the interface is collaborative. Define the variational problem for the functional

$$J_{\beta,\Lambda}(u,\Omega_1,\Omega_2) = \int_\Omega |\nabla u|^2\,dx + \beta\int_{\partial^*\Omega_1\cap\partial^*\Omega_2} u^2\,d\mathcal{H}^{d-1} + \Lambda|\{u>0\}|$$

in an appropriate class. We fix the boundary data for Ω_1, Ω_2 and g. Precisely, let

- E_1 and E_2 be two smooth, bounded and disjoint sets of positive distance in $\mathbb{R}^d$;
- $D := \mathbb{R}^d \setminus \left(\overline{E}_1 \cup \overline{E}_2\right)$;
- $\Omega_i = E_i$ in $\mathbb{R}^d \setminus D$;
- $g \in H^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be a non-negative function such that

$$g \equiv 1 \quad \text{on} \quad E_1 \cup E_2\,.$$

We define the following admissible set of functions

$$\mathcal{V} = \left\{ u \in H^1(\mathbb{R}^d) \; : \; u \geq 0 \text{ in } \mathbb{R}^d \text{ and } u - g \in H^1_0(D) \right\}.$$

Then, fixed $u \in \mathcal{V}$, we define the admissible set $\mathcal{A}(u)$ as the set of all couples (Ω_1, Ω_2) of Lebesgue measurable sets such that:

- $\Omega_1 \cap \Omega_2 = \emptyset$, $E_1 \subset \Omega_1$ and $E_2 \subset \Omega_2$ Lebesgue almost-everywhere;
- Ω_1 and Ω_2 have finite perimeter (as subsets of $\mathbb{R}^d$);
- $\{u > 0\} \subset \Omega_1 \cup \Omega_2$ Lebesgue almost-everywhere.

Thus, we interested in the problem

$$\min \left\{ J_{\beta,\Lambda}(u, \Omega_1, \Omega_2) \; : \; u \in \mathcal{V}, (\Omega_1, \Omega_2) \in \mathcal{A}(u) \right\}. \tag{2.1}$$

Main Theorem

Theorem 2

Let D be a smooth bounded open set in $\mathbb{R}^2$. Given sets E_1 and E_2, and a function g as above, there are a function $u \in \mathcal{V}$ and sets $(\Omega_1, \Omega_2) \in \mathcal{A}(u)$ that solve the variational problem (2.1).
Conversely, if (u, Ω_1, Ω_2) is a solution to (2.1), then also $(u, \widetilde{\Omega}_1, \widetilde{\Omega}_2)$ is a solution to (2.1), where

$$\widetilde{\Omega}_1 = \{u > 0\} \cap \Omega_1 \qquad \text{and} \qquad \widetilde{\Omega}_2 = \{u > 0\} \cap \Omega_2.$$

Moreover,

1. *the boundary $\partial\{u > 0\}$ is $C^{1,\alpha}$-regular in D;*
2. *the interface $\partial\Omega_1 \cap \partial\Omega_2$ is C^∞ in the open set $D \cap \{u > 0\}$ and is C^1 regular up to the boundary $D \cap \partial\{u > 0\}$. Moreover, $\partial\Omega_1 \cap \partial\Omega_2$ reaches $\partial\{u > 0\}$ orthogonally.*

Differences with competitive two phases

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This problem is closely related to the Alt-Caffarelli functional and in fact it can be seen as a two phase variant of it.
Notice that if (u_1, Ω_1) and (u_2, Ω_2) are two minimizers of the one-phase Bernoulli functional $J_{\sqrt{\Lambda}}$ with disjoint supports $(\Omega_1 \cap \Omega_2 = \emptyset)$, the triple $(\Omega_1,\ \Omega_2,\ u = u_1 + u_2)$ might not be a minimizer of $J_{\beta,\Lambda}$, even if the Hausdorff distance between Ω_1 and Ω_2 is strictly positive. In fact, it might be convenient to enlarge the domains Ω_1 and Ω_2 in order to obtain a non-empty interface $\partial\Omega_1 \cap \partial\Omega_2$ that will allow to have competitors which are not vanishing identically on the entire free boundaries $\partial\Omega_1$ and $\partial\Omega_2$ as it can be seen by an explicit example.

Existence

As before, to find existence is not an easy task. One of the main difficulty comes from the fact that for a minimizing sequence $(u_n, \Omega_n^1, \Omega_n^2)$ we do not have much of control on the perimeter of the sets Ω_n^1, Ω_n^2. To overcome this difficulty we introduce once again a family of approximating problems which are easier to solve.

Existence

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$$\min \Big\{ J_\varepsilon\big(u, \Omega_1, \Omega_2\big) \ : u \in \mathcal{V}, \big(\Omega_1, \Omega_2\big) \in \mathcal{A}(u) \Big\}, \tag{2.2}$$

where the functional J_ε is defined as

$$\begin{aligned} J_\varepsilon(u, \Omega_1, \Omega_2) := \int_D |\nabla u|^2 \, dx + \Lambda |\{u > 0\} \cap D| \\ + \frac{\beta}{2} \left(\int_{\partial^* \Omega_1} u^2 \, d\mathcal{H}^{d-1} + \int_{\partial^* \Omega_2} u^2 \, d\mathcal{H}^{d-1} \right) \\ + \varepsilon \Big(\mathrm{Per}(\Omega_1) + \mathrm{Per}(\Omega_2) \Big). \end{aligned}$$

Sketch of the proof

The proof is made in several steps:

1. For $\varepsilon > 0$ prove existence of a solution $(u^{\varepsilon}, \Omega_1^{e}, \Omega_2^{\varepsilon})$ for J_e
2. Prove global Lipschitz regularity for u_e and $C^{1,\alpha}$ estimate for Ω_1^{ε}(theory of almost minimizers of the perimeter);
3. Send $\varepsilon \to 0$ and construct a candidate solution for the problem $(u^0, \Omega_1^0, \Omega_2^0)$ showing that it is optimal in some sense ;
4. Shot that $(u^0, \Omega_1^0, \Omega_2^0) \in \mathcal{A} \times \mathcal{V}(u)$
5. Use a blow up argument to prove regularity of the boundary.

Construction of u

From direct methods in Calculus of Variations and Ascoli Arzerlà theorem we find $u \in H^1(\mathbb{R}^d)$ such that

1. for every fixed $\delta > 0$, $u_{\varepsilon_n} \to u$ uniformly in D_δ as $n \to \infty$;
2. $u_{\varepsilon_n} \to u$ strongly in $L^2(\mathbb{R}^d)$ and pointwise almost-everywhere in $\mathbb{R}^d$;
3. $\nabla u_{\varepsilon_n} \to \nabla u$ weakly in $L^2(\mathbb{R}^d)$.

By construction, we have $u - g \in H_0^1(D)$ and

$$u \in H_0^1((\mathbb{R}^n) \qquad \text{and} \qquad u \in C^{0,\frac{1}{3}}(\overline{D}_\delta) \quad \text{for every} \quad \delta > 0.$$

Moreover,

$$0 \leq u \leq \|g\|_{L^\infty(\mathbb{R}^d)} \qquad \text{and} \qquad \Delta u \geq 0 \quad \text{in} \quad D.$$

Construction of Ω_i^0

Choose a ball

$$B_R(x_0) \subset\subset D \cap \{u > 0\}.$$

Then, there are $t > 0$ and $\delta > 0$ such that

$$\overline{B}_R(x_0) \subset D_\delta \cap \{u \geq t\}.$$

By the uniform convergence of u_{ε_n} to u on D_δ

$$u_{\varepsilon_n} \geq \frac{t}{2} \quad \text{in} \quad \overline{B}_R(x_0). \qquad \text{for } n \text{ large}$$

Using this inequality and the optimality of $\left(u_{\varepsilon_n}, \Omega^1_{\varepsilon_n}, \Omega^2_{\varepsilon_n}\right)$

$$\Omega^1_{\varepsilon_n} \cap B_R(x_0) \qquad \text{and} \qquad \Omega^2_{\varepsilon_n} \cap B_R(x_0)$$

have uniformly bounded perimeter. In particular, up to a subsequence there are sets of finite perimeter and such that,

$$\chi_{\Omega^1_{\varepsilon_n} \cap B_R(x_0)} \to \chi_{\Omega^1_{R,x_0}} \qquad \text{and} \qquad \chi_{\Omega^2_{\varepsilon_n} \cap B_R(x_0)} \to \chi_{\Omega^2_{R,x_0}},$$

Construction of Ω_1^0

part two

Thus, by a diagonal sequence argument, we can define the sets Ω_1 and Ω_2 as the union of Ω^1_{R,x_0} and Ω^2_{R,x_0} over all balls

$$B_R(x_0) \subset\subset D \cap \{u > 0\},$$

of radius $R \in \mathbb{Q}$ and center with rational coordinates $x_0 \in \mathbb{Q}^d$,

$$\Omega_i := E_i \cup \bigcup_{R,x_0} \Omega^i_{R,x_0} \qquad \text{for} \quad i = 1, 2.$$

By construction, Ω_1 and Ω_2 have locally finite perimeter in $D \cap \{u > 0\}$ and satisfy

$$E_i \subset \Omega_i \subset \Big(\big(E_1 \cup E_2 \big) \cap B_\rho \Big),$$

and the sets Ω_1 and Ω_2 are disjoint, $|\Omega_1 \cap \Omega_2| = 0$.

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$$E_i \subset \Omega_i \subset \Big((E_1 \cup E_2) \cap B_\rho \Big),$$

and the sets Ω_1 and Ω_2 are disjoint, $|\Omega_1 \cap \Omega_2| = 0$. Notice that we do not have a priori that Ω_1 and Ω_2 have finite perimeter in $\mathbb{R}^d$, so at this stage they might not be in the admissible class $\mathcal{A}(u)$.

The key observation to solve the existence probelm is to show that u is an almost-minimizer of the classical one-phase functional of Alt and Caffarell, which immediately gives that $\{u > 0\}$ has locally finite perimeter in D. To prove that Ω_1, Ω_2 have finite perimeter we need to prove a non degenereacy result.

Proposition 5 (Non-collapsing)

Let $u \in H^1(\mathbb{R}^d)$ be the limit function constructed before. Then, there is a positive constant $t > 0$ such that $u \geq t$ in a neighborhood of $\overline{E}_1 \cup \overline{E}_2$.

The proof of this fact is based on the following

Lemma 6 (Density estimate)

There is a constant $c > 0$ and $R_0 > 0$ such that

$$\big|B_R(x_0) \cap \{u = 0\}\big| \geq c|B_R| \quad \text{for every} \quad B_R(x_0) \subset D \quad \text{with} \quad u(x_0) = 0.$$

Thank you for your attention